FROM FILTERS TO WAVELETS VIA DIRECT LIMITS

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Abstract. We present a new proof of a theorem of Mallat which describes a construction of wavelets starting from a quadrature mirror filter. Our main innovation is to show how the scaling function associated to the filter can be used to identify a certain direct limit of Hilbert spaces with $L^2(\mathbb{R})$ in such a way that one can immediately identify the wavelet basis. Our arguments also use a pair of isometries introduced by Bratteli and Jorgensen, and exploit the geometry inherent in the Cuntz relations satisfied by these isometries.

A *wavelet* is a function $\psi \in L^2(\mathbb{R})$ such that

$$\{\psi_{j,k} : x \mapsto 2^{-j/2}\psi(2^{-j}x - k) : j, k \in Z\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. There are, remarkably, many different wavelets, and they have proved to be enormously useful in both theory and applications. So there has been a great deal of interest in methods of constructing wavelets. One famous construction of Mallat [7] starts from a *quadrature mirror filter*: a function $m_0 : \mathbb{T} \to \mathbb{C}$ such that

$$|m_0(z)|^2 + |m_0(-z)|^2 = 1 \text{ for every } z \in \mathbb{T}.$$

Our goal here is to present a new proof of Mallat's theorem based on the concept of a direct limit.

Mallat proved his theorem in two stages. From the filter m_0 he built a *multiresolution analysis*, in which a central role is played by a *scaling function* $\phi \in L^2(\mathbb{R})$ satisfying $\phi(2x) = m_0(e^{2\pi ix})\phi(x)$ [7, Theorem 2]. He then used what he described as a "by now classic" algorithm to generate the wavelet [7, §4]. Mallat's construction has since been refined and discussed in several books. For example, [5, §5.3–4] contains a relatively elementary proof of his theorem, in which some of the analysis has been simplified but the overall strategy is that of Mallat. In our proof, the scaling function still plays a central role: we use it to identify a certain direct limit with $L^2(\mathbb{R})$, and the existence of the wavelet then follows almost immediately from the geometry implicit in some operator-theoretic equations called the *Cuntz relations*. The analytic content of our proof is much the same as that in the standard sources, and we refer to them for details, but our organisation seems to be quite different.

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Our arguments seem to be more natural in the Fourier or frequency domain, so we work there throughout, and our construction yields the Fourier transform of the wavelet. One effect of working in the Fourier domain is that the scaling equation (Equation (4) below) involves multiplication rather than convolution. In the interests of clarity, we shall consider only the classical (dyadic) wavelets, but we are optimistic that our approach will also shed light in other situations where wavelet bases are used.

Throughout, m_0 will be a quadrature mirror filter such that m_0 is smooth at 1, $m_0(1) = 1$ and $m_0(z) \neq 0$ for z in the right half-circle. We define $m_1 : \mathbb{T} \to \mathbb{C}$ by

$$m_1(z) := z\overline{m_0(-z)}.$$

We now define two operators S_0 and S_1 on $L^2(\mathbb{T})$ by

$$(S_i f)(z) := 2^{1/2} m_i(z) f(z^2). \tag{1}$$

Our starting point is the following observation of Bratteli and Jorgensen [2]:

Lemma 1. *The operators S_i satisfy $S_0^* S_0 = 1 = S_1^* S_1$ and $S_0 S_0^* + S_1 S_1^* = 1$.*

To prove this, first verify that the adjoints S_i^* are given by

$$(S_i^* f)(e^{2\pi i x}) = 2^{-1/2}\big(\overline{m_i(e^{\pi i x})} f(e^{\pi i x}) + \overline{m_i(-e^{\pi i x})} f(-e^{\pi i x})\big),$$

and then compute $S_i^* S_i$ and $S_0 S_0^* + S_1 S_1^*$.

The formal computation in Lemma 1 has some very interesting geometric consequences. The relations $S_i^* S_i = 1$ say that the operators S_i are isometries of $L^2(\mathbb{T})$ into itself, and imply that the operators $S_i S_i^*$ are the orthogonal projections onto the ranges of the S_i (which are automatically closed because the S_i are isometries). Since the sum of two projections is a projection only when their ranges are orthogonal, the *Cuntz relation* $S_0 S_0^* + S_1 S_1^* = 1$ implies that the ranges $S_i L^2(\mathbb{T})$ of the S_i are orthogonal complements of each other, so that

$$L^2(\mathbb{T}) = S_0 L^2(\mathbb{T}) \oplus S_1 L^2(\mathbb{T}). \tag{2}$$

We now recall a construction from algebra. Suppose that we have Hilbert spaces H_n and isometries $T_n : H_n \to H_{n+1}$ for all $n \in \mathbb{N}$. The *direct limit* (H_∞, U_n) consists of a Hilbert space H_∞ and isometries $U_n : H_n \to H_\infty$ which satisfy $U_{n+1} \circ T_n = U_n$ and which have the following universal property: for every family of isometries $\{R_n\}$ of H_n into a Hilbert space K such that $R_{n+1} \circ T_n = R_n$, there is a unique isometry $R_\infty : H_\infty \to K$ such that $R_\infty \circ T_n = R_n$ for every n. We illustrate this universal

property with the diagram:

$$
\begin{tikzcd}
H_0 \arrow[r, "T_0"'] \arrow[rrrr, bend left, "U_0"] \arrow[dr, "R_0"'] & H_1 \arrow[r, "T_1"'] \arrow[rrr, bend left, "U_1"] \arrow[d, "R_1"] & H_2 \arrow[r, "T_2"'] \arrow[rr, bend left] \arrow[dl, "R_2"] & \cdots & H_\infty \arrow[dlll, dashed, "R_\infty"] \\
& K & & &
\end{tikzcd}
$$

The uniqueness implies that $H_\infty = \overline{\bigcup_{n=1}^\infty U_n H_n}$, since otherwise we could define R_∞ arbitrarily on $(\bigcup U_n H_n)^\perp$.

It is not hard to see that the direct limit exists. Indeed, identifying each $h \in H_n$ with all its images $T_{m-1}T_{m-2}\cdots T_n h \in H_m$ defines an equivalence relation $\sim$ on the disjoint union $\bigsqcup H_n$, and because the T_n are isometries and hence inner-product preserving, the set $H' := (\bigsqcup H_n)/\sim$ of equivalence classes is naturally an inner-product space; completing H' gives a Hilbert space H_∞, and the maps U_n which send elements of H_n to their class in $H' \subset H_\infty$ have the required properties. However, the important point is that the construction does not matter, since the universal property identifies the direct limit up to isomorphism: to identify H_∞ with a Hilbert space K, for example, we just need to find isometries $R_n : H_n \to K$ as above such that $K = \overline{\bigcup_{n=1}^\infty R_n H_n}$, and then R_∞ is an isomorphism of H_∞ onto K.

If we start with a single isometry S on a Hilbert space H, we can take the direct limit (H_∞, U_n) of the system in which every H_n is H and every T_n is S. Now consider the diagram

$$
\begin{tikzcd}
H \arrow[r, "S"] \arrow[d, "S"] & H \arrow[r, "S"] \arrow[d, "S"] & H \arrow[r, "S"] \arrow[d, "S"] & \cdots & \cdots \arrow[r] & H_\infty \arrow[d, dashed, bend left, "S_\infty"] \\
H \arrow[r, "S"] \arrow[ur, "1"] & H \arrow[r, "S"] \arrow[ur, "1"] & H \arrow[r, "S"] \arrow[ur, "1"] & \cdots & \cdots \arrow[r] & H_\infty, \arrow[u, dashed, bend left, "1_\infty"]
\end{tikzcd}
$$

where the horizontal arrows going into H_∞ are there to remind us that we have isometries U_n of every copy H_n of H into H_∞. Applying the universal property of the top row to the downward arrows (or more properly, to $R_n := U_n \circ S$) gives an isometry S_∞ on H_∞ which is characterised by $S_\infty(U_n h) = U_n(Sh)$. Similarly, applying the universal property of the bottom row to the NE arrows gives an isometry 1_∞ which is characterised by $1_\infty(U_n h) = U_{n+1}h$. Then for every n and every $h \in H$ we have

$$
\begin{aligned}
&(1_\infty S_\infty)(U_n h) = 1_\infty(S_\infty U_n h) = 1_\infty(U_n(Sh)) = U_{n+1}(Sh) = U_n h \quad \text{and} \\
&(S_\infty 1_\infty)(U_n h) = S_\infty(U_{n+1}h) = U_{n+1}(Sh) = U_n h,
\end{aligned}
$$

which implies that 1_∞ is an inverse for S_∞. This process of passing to the direct limit, therefore, turns the isometry S into a unitary S_∞. Since

$$S_\infty(U_{n+1}h) = U_{n+1}(Sh) = U_n h, \tag{3}$$

this unitary is an isomorphism of the copy $U_{n+1}H$ of $H_{n+1} = H$ onto the copy U_nH of $H_n = H$.

Applying the process described in the previous paragraph to the isometry S_0 on $L^2(\mathbb{T})$ defined in (1) gives a Hilbert space H_∞ and a unitary operator S_∞ on H_∞. Our next task is to identify the direct limit H_∞ with $L^2(\mathbb{R})$. Mallat proved that, under our hypotheses on m_0, there is a *scaling function*[1] $\phi \in L^2(\mathbb{R})$ of norm 1 such that

$$\phi(2x) = m_0(e^{2\pi i x})\phi(x) \quad \text{and} \tag{4}$$

$$\sum_{k\in\mathbb{Z}} |\phi(x+k)|^2 = 1 \tag{5}$$

for every $x \in \mathbb{R}$. Indeed, he proved that the infinite product

$$\phi(x) = \prod_{n=1}^{\infty} m_0(\exp(2\pi i 2^{-n}x))$$

has the required properties (see [7, pages 76–77] or [5, §5.4]). For example, if m_0 is the characteristic function of the right half-circle, then $\phi = \chi_{[-1/2,1/2]}$ is a scaling function. (In [6, page 225] this is proved under the milder hypothesis that m_0 satisfies *Cohen's condition*, which is a necessary and sufficient condition for the existence of such a function ϕ.)

We now fix $n \in \mathbb{Z}$, and define $R_n : H_n = L^2(\mathbb{T}) \to L^2(\mathbb{R})$ by

$$(R_n f)(x) = 2^{-n/2} f(\exp(2\pi i 2^{-n}x))\phi(2^{-n}x).$$

Each R_n is an isometry: indeed, a change of variables and an application of the monotone convergence theorem shows that

$$\begin{aligned}
\|R_n f\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} 2^{-n} |f(\exp(2\pi i 2^{-n}x))\phi(2^{-n}x)|^2\,dx \\
&= \int_{\mathbb{R}} |f(e^{2\pi i x})\phi(x)|^2\,dx \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 |f(e^{2\pi i t})\phi(t+n)|^2\,dt \\
&= \int_0^1 |f(e^{2\pi i t})|^2 \Big(\sum_{n\in\mathbb{Z}} |\phi(t+n)|^2\Big)\,dt,
\end{aligned}$$

which by (5) is precisely the norm of f in $L^2(\mathbb{T})$. The identity (4) implies that the isometries R_n are compatible with the isometries $S_0 : H_n = L^2(\mathbb{T}) \to H_{n+1} = L^2(\mathbb{T})$

[1] In the literature, it is the inverse Fourier transform $\check{\phi}$ of this function ϕ which is usually called a scaling function.

in the direct system:

$$\begin{aligned}R_{n+1}(S_0 f)(x) &= 2^{-(n+1)/2}2^{1/2}m_0(\exp(2\pi i 2^{-(n+1)}x))f(\exp(2\pi i 2^{-(n+1)}x)^2)\phi(2^{-(n+1)}x)\\ &= 2^{-n/2}f(\exp(4\pi i 2^{-(n+1)}x))\phi(2(2^{-(n+1)}x))\\ &= (R_n f)(x).\end{aligned}$$

Thus the universal property of the direct limit gives an isometry $R_\infty : H_\infty \to L^2(\mathbb{R})$. Notice that if we set $V_n := \operatorname{range} R_n = R_\infty(U_n L^2(\mathbb{T}))$, then the last calculation shows that $V_n \subset V_{n+1}$.

Lemma 2. *The isometry R_∞ intertwines the unitary S_∞ on H_∞ and the dilation operator D on $L^2(\mathbb{R})$ defined by $(D\xi)(x) = 2^{1/2}\xi(2x)$.*

Proof. We let $f \in H_{n+1} = L^2(\mathbb{T})$, and compute:

$$\begin{aligned}D(R_\infty(U_{n+1}f))(x) &= D(R_{n+1}f)(x)\\ &= 2^{1/2}(R_{n+1}f)(2x)\\ &= 2^{1/2}2^{-(n+1)/2}f(\exp(2\pi i 2^{-(n+1)}2x))\phi(2^{-(n+1)}2x)\\ &= (R_n f)(x)\\ &= R_\infty(U_n f)(x).\end{aligned}$$

Since we know from (3) that $S_\infty \circ U_{n+1} = U_n$, this implies that $D \circ R_\infty$ and $R_\infty \circ S_\infty$ agree on the range of every U_{n+1}, and hence on H_∞. □

Since S_∞ is an isomorphism of $U_{n+1}H$ onto $U_n H$, it follows from Lemma 2 that[2] $DV_{n+1} = V_n$ for every $n \in \mathbb{Z}$. The following lemma is proved in [7, pages 78–79] and [5, Lemmas 5.47 and 5.48]. (A different argument which proves the analogous property of the inverse Fourier transforms is given in Propositions 5.3.1 and 5.3.2 of [3].)

Lemma 3. *We have $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$ and $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$.*

We write W_n for the complement $V_{n+1} \ominus V_n$ of V_n in V_{n+1}. Then the subspaces W_n are mutually orthogonal, and it follows from Lemma 3 that $L^2(\mathbb{R})$ decomposes as the direct sum $\bigoplus_{n\in\mathbb{Z}} W_n$. Since $D^{-1}V_{n-1} = V_n$ for every n, we have $DW_{n+1} = W_n$ for every n. Thus to find an orthonormal basis for $L^2(\mathbb{R})$, it suffices to find an orthonormal basis for one W_n, and then this together with all its dilates will be an orthonormal basis for $L^2(\mathbb{R})$.

So we seek an orthonormal basis for W_0. With $H = L^2(\mathbb{T})$, we have

$$W_0 = V_1 \ominus V_0 = R_1 H \ominus R_0 H = R_1 H \ominus R_1(S_0 H) = R_1(H \ominus S_0 H).$$

At this point we recall from (2) that the complement of $S_0 H$ is the range $S_1 H$ of the other isometry S_1. Since $R_1 S_1$ is an isometry, it maps the usual orthonormal basis

[2] This formula looks slightly different from the usual dilation property of a multiresolution analysis because we are working in the frequency domain; in [7] and [5], for example, the space denoted by V_n is the inverse Fourier transform of our V_n.

$\{e_k : z \mapsto z^k : k \in \mathbb{Z}\}$ for $L^2(\mathbb{T})$ into an orthonormal basis for W_0. We deduce that the functions

$$\psi_k(x) = R_1 S_1(e_{-k})(x) = 2^{-1/2}\big(2^{1/2} m_1(\exp(2\pi i 2^{-1}x))e^{-2\pi i k x}\big)\phi(2^{-1}x)$$

form an orthonormal basis for W_0. We set

$$\psi(x) := m_1(e^{\pi i x})\phi(2^{-1}x),$$

so that the basis elements take the form

$$\psi_k(x) = e^{-2\pi i k x}\psi(x).$$

If we now define

$$\psi_{j,k}(x) = (D^j\psi_k)(x) = 2^{j/2}\exp(-2\pi i k 2^j x)\psi(2^j x),$$

then $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Since the inverse Fourier transform intertwines D and D^{-1}, and intertwines multiplication by $e^{2\pi i k x}$ and the translation operator which takes ξ to $\xi(\cdot + k)$, the functions

$$\check{\psi}_{j,k}(x) = 2^{-j/2}\check{\psi}(2^{-j}x - k)$$

also form an orthonormal basis for $L^2(\mathbb{R})$. In other words, $\check{\psi}$ is a wavelet, and we have proved Mallat's theorem:

Theorem 4. *Suppose that m_0 is a quadrature mirror filter such that m_0 is smooth at 1, $m_0(1) = 1$ and $m_0(z) \neq 0$ for z in the right half-circle, and let ϕ be a function satisfying the scaling conditions* (4) *and* (5). *Define $\psi : \mathbb{R} \to \mathbb{C}$ by*

$$\psi(x) = e^{\pi i x}\,\overline{m_0(-e^{\pi i x})}\,\phi(x/2).$$

Then the inverse Fourier transform $\check{\psi}$ is a wavelet.

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